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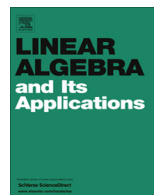
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# Linear Algebra and its Applications

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## Compressions of maximal dissipative and self-adjoint linear relations and of dilations<sup>☆,☆☆</sup>



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### ABSTRACT

In this paper we generalize results from Stenger (1968) [30], Nudelman (2011) [28] and Azizov and Dijksma (2012) [7] to maximal dissipative and self-adjoint linear relations and discuss related results for nonnegative self-adjoint extensions of nonnegative symmetric linear relations and self-adjoint dilations of maximal dissipative linear relations.

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## 1. Introduction

Stenger [30] proved that the compression of a self-adjoint operator in a Hilbert space to a subspace with finite codimension is self-adjoint in this subspace. This result was generalized by Nudelman [28] who proved that the compression of a densely defined maximal dissipative operator in a Hilbert space to a subspace with finite codimension is densely defined and maximal dissipative in this subspace. Azizov and Dijksma [7] showed that converses of these statements also hold: If an operator in a Hilbert space is densely defined and symmetric (dissipative) and its compression to a subspace with finite codimension is self-adjoint (maximal dissipative) then the operator is self-adjoint (maximal dissipative). In this paper we prove analogous results for linear relations in Hilbert and Krein spaces; thus we remove the condition that the operators are densely defined and allow them to be “multi-valued operators,” that is, linear relations. We assume that the reader is familiar with linear relations as in for example [9,16,27,11,20]. We use only elementary facts from operator theory in Krein spaces and they can be found in [8,21,6]. Since we aim at a reasonably self-contained presentation, we partially repeat some facts and proofs.

As an application we give sufficient conditions under which the compression of the hard (or Friedrichs) and the soft extension of a closed nonnegative symmetric linear relation  $S$  in a Hilbert space to a subspace with finite codimension is the hard and soft extension of the compression of  $S$  to this subspace. One of these conditions for the hard extension is that  $S$  is densely defined. This condition alone is not sufficient for the soft extension of  $S$ . We give a counter example. For details about nonnegative self-adjoint extensions of a densely defined nonnegative symmetric operator we refer to [23], [29, Sections 124 and 125], [1, Section 109], and for such extensions of a nonnegative symmetric linear relation to [11].

As a second application we prove that the minimal self-adjoint dilation of the compression of a maximal dissipative linear relation in a Hilbert space to a subspace of finite codimension is a compression of the minimal self-adjoint dilation of this maximal dissipative linear relation. But first we show that minimal self-adjoint dilations exist for maximal dissipative linear relations. For densely defined maximal dissipative operators this is well-known, see for example [26, Chapter 4].

In the sequel, if  $T$  is a linear relation in a Krein space  $\mathcal{K}$ , then we denote by  $\rho(T)$  the resolvent set of  $T$ , by  $R_T(\lambda) = (T - \lambda)^{-1}$ ,  $\lambda \in \rho(T)$ , the resolvent of  $T$  and by  $K_T(\lambda, \mu)$  the kernel

$$K_T(\lambda, \mu) = \frac{R_T(\lambda) - R_T(\mu)^*}{\lambda - \mu^*} - R_T(\mu)^* R_T(\lambda), \quad \lambda, \mu \in \rho(T), \quad \lambda \neq \mu^*.$$

The kernel was investigated in [24] in connection with generalized resolvents and used to prove the converse of part of the main result of that paper (Theorem 5.1; see p. 222). It was further studied in [12, Section 2], [14, Proposition 2.1], [5], [3].

We briefly describe the contents of the four sections and the Appendix which come after this introduction. In Section 2 we recall from [24] and reprove some basic facts for a maximal dissipative linear relation  $T$  related to  $\rho(T)$ , the decomposition  $T = T_{\text{op}} \dot{+} T_{\infty}$  of  $T$  into its multi-valued part  $T_{\infty} = \{0\} \times T(0)$  and its operator part  $T_{\text{op}} = T \ominus T_{\infty}$  and the nonnegativity of  $K_T(\lambda, \mu)$  on  $\mathbb{C}_-$ . In Section 3 and the beginning of Section 4 we prove the theorems concerning the compression of a maximal dissipative relation  $T$  and the compression of a self-adjoint relation  $S$ , see Theorem 3.3 (and Theorem 3.1) and Theorem 4.1. They are the generalizations of the theorems of Nudelman, Stenger and Azizov and Dijksma described at the beginning of this introduction. New in the theorems is the appearance of the equalities  $T(0) = T^*(0)$  and  $S(0) = S^*(0)$ . They are automatically satisfied when  $T$  and  $S$  are densely defined and then the theorems reduce to [7, Theorem 3.4]. In the second part of Section 4 we apply Theorem 4.1 to the hard and soft extensions of a nonnegative symmetric relation in a Hilbert space. Here the main theorems are Theorem 4.5 and Theorem 4.7. In Section 5 we prove the existence of a self-adjoint dilation of a maximal dissipative relation  $T$  and show that the dilation can be chosen minimal in which case it is essentially unique, see Theorem 5.1. In the proof we use the reproducing kernel Hilbert space associated with the kernel  $K_T(\lambda, \mu)$  on  $\mathbb{C}_-$ . Theorem 5.3 states that, roughly formulated, “the compression of the minimal dilation is the minimal dilation of the compression.” Finally, in the Appendix we explain a vector notation which we occasionally use in the sequel.

By  $P_{\mathcal{G}}$  we denote the orthogonal projection in the Krein space  $\mathcal{K}$  onto the non-degenerated subspace  $\mathcal{G}$  of  $\mathcal{K}$ . By  $\dot{+}$  we mean the sum in  $\mathcal{K}^2$ .

## 2. Dissipative linear relations in a Hilbert space

In this paper a linear relation  $T$  in a Hilbert space or Krein space  $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$  is called *dissipative* if  $\operatorname{Im} (g, f)_{\mathcal{H}} \geq 0$ ,  $\{f, g\} \in T$ . It is *maximal dissipative* if it is not properly contained in another dissipative linear relation in  $\mathcal{H}$ . Maximal dissipative linear relations were introduced and studied in [24]. We repeat and reprove some of their basic properties. Item (iii) of Lemma 2.1 and item (iv) of Lemma 2.2 below were proved in [24, Section 4] via the Cayley transform. The equivalences in Lemma 2.3 are at least implicitly contained in [24, Formula (5.23) and Lemma 4.1].

**Lemma 2.1.** *For a linear relation  $T$  in a Hilbert space  $\mathcal{H}$  the following statements hold.*

- (i) *If  $T$  is maximal dissipative, then  $T$  is closed.*
- (ii) *If  $T$  is dissipative, then  $\operatorname{dom} T \subset T(0)^{\perp}$ , in particular: if moreover  $T$  is densely defined, then it is an operator.*
- (iii) *If  $T$  is maximal dissipative, then  $\overline{\operatorname{dom} T} = T(0)^{\perp}$ , in particular: in this case  $T$  is an operator if and only if it is densely defined.*

For any closed linear relation  $T$  in a Hilbert or Krein space, the equality  $\overline{\operatorname{dom} T} = T(0)^{\perp}$  (as in (iii) of the above lemma) is equivalent to the equality  $T^*(0) = T(0)$ , because  $T^*(0) = (\operatorname{dom} T)^{\perp}$ .

**Proof of Lemma 2.1.** If  $T$  is dissipative, then so is its closure. This implies (i). To prove (ii) and (iii), let  $T$  be a dissipative linear relation in  $\mathcal{H}$ . Consider  $v \in T(0)$  and  $x \in \operatorname{dom} T$ . Then there is a  $y \in \mathcal{H}$  such that  $\{x, y + \alpha v\} \in T$  for all  $\alpha \in \mathbb{C}$ , and hence

$$\operatorname{Im} (\alpha v, x)_{\mathcal{H}} \geq -\operatorname{Im} (y, x)_{\mathcal{H}}, \quad \alpha \in \mathbb{C}.$$

This implies  $(v, x)_{\mathcal{H}} = 0$ , whence (ii). We now prove (iii). Let  $v$  be an element in  $T(0)^{\perp} \ominus \operatorname{dom} T$ . Then  $T_{\text{ext}} := T \dot{+} \operatorname{span}\{0, v\}$  is a dissipative linear relation which extends  $T$ . Since  $T$  is maximal dissipative,  $T_{\text{ext}} = T$  and hence  $\{0, v\} \in T$ . It follows that  $v \in T(0)^{\perp} \cap T(0) = \{0\}$  and this implies (iii).  $\square$

**Lemma 2.2.** *For a linear relation  $T$  in a Hilbert space  $\mathcal{H}$  the following statements are equivalent.*

- (1)  *$T$  is maximal dissipative.*
- (2)  *$T$  is dissipative and  $\rho(T) \cap \mathbb{C}_- \neq \emptyset$ .*
- (3)  *$T$  is dissipative and  $\mathbb{C}_- \subset \rho(T)$ .*
- (4)  *$T(0)$  is closed and  $T = T_{\text{op}} \dot{+} T_{\infty}$ , where  $T_{\infty} := \{0\} \times T(0)$  and  $T_{\text{op}} := T \ominus T_{\infty}$  (in  $\mathcal{H}^2$ ) is a maximal dissipative operator in the Hilbert space  $T(0)^{\perp}$ .*

If  $T$  has these properties and  $P_0$  is the orthogonal projection in  $\mathcal{H}$  onto  $T(0)^{\perp}$ , then

$$\lim_{y \in \mathbb{R}, y \rightarrow -\infty} -iyR_T(iy)u = P_0u, \quad u \in \mathcal{H}.$$

$T_{\text{op}}$  and  $T_{\infty}$  in item (4) of the lemma are called the operator and the multivalued part of  $T$ .

**Proof of Lemma 2.2.** Assume first that  $T$  is densely defined. Then the equivalence between items (1), (2) and (3) follows from [6, Corollary 2.2.5 and Lemma 2.2.8], see also [22, Section V.3.10]. Item (4) trivially coincides with item (1) since  $T(0) = \{0\}$ . In this case  $P_0 = I$  and the last statement follows from [22, Problem V.3.33].

Now we drop the assumption that  $T$  is densely defined. We prove the equivalence between (1) and (4). First assume that  $T$  is decomposed as in (4). Let  $\widehat{T}$  be a dissipative extension of  $T$  in  $\mathcal{H}$ . Then

$$(T_{\text{op}} + i)^{-1} \dot{+} T_{\infty}^{-1} = (T + i)^{-1} \subset (\widehat{T} + i)^{-1}.$$

By item (3) applied to  $T_{\text{op}}$ , the equality implies that  $(T + i)^{-1}$  is an everywhere defined operator. We claim that  $(\widehat{T} + i)^{-1}$  is an operator. If the claim is true, then  $(T + i)^{-1} = (\widehat{T} + i)^{-1}$ , hence  $\widehat{T} = T$  and (1) holds. It remains to prove the claim. Let  $\{0, x\} \in (\widehat{T} + i)^{-1}$ . Then  $\{x, -ix\} \in \widehat{T}$  and, as  $\widehat{T}$  is dissipative,

$$0 \leq \operatorname{Im}(-ix, x)_{\mathcal{H}} = -(x, x)_{\mathcal{H}} \leq 0.$$

This proves  $x = 0$  and the claim.

Now assume (1). Then  $T$  is closed and hence  $T(0)$  is closed. Decompose  $T$ :  $T = S \dot{+} T_{\infty}$  with  $S := T \ominus T_{\infty}$ , the orthogonal difference in  $\mathcal{H}^2$ . By Lemma 2.1,  $S$  is a dissipative operator in  $T(0)^{\perp}$ . It is maximal dissipative because every dissipative extension of  $S$  in  $T(0)^{\perp}$  gives rise to an extension of  $T$ . Thus  $S = T_{\text{op}}$  and (4) holds.

The proof of the remaining equivalences and the last statement in the lemma can be given by applying the corresponding results for the densely defined operator  $T_{\text{op}}$ . We omit the details.  $\square$

**Lemma 2.3.** *For a linear relation  $T$  in a Hilbert space  $\mathcal{H}$  the following statements are equivalent.*

- (a)  $T$  is maximal dissipative.
- (b) There is a  $\lambda \in \mathbb{C}_- \cap \rho(T)$  such that  $K_T(\lambda, \lambda) \geq 0$ .
- (c)  $\mathbb{C}_- \subset \rho(T)$  and  $K_T(\lambda, \lambda) \geq 0$  for all  $\lambda \in \mathbb{C}_-$ .
- (d)  $\mathbb{C}_- \subset \rho(T)$  and  $K_T(\lambda, \mu)$  is a nonnegative kernel on  $\mathbb{C}_-$ .

**Proof of Lemma 2.3.** The equivalence between items (a), (b) and (c) is noted in [16, Proposition 3.4(i)]. It directly follows from the fact that for each  $\lambda \in \rho(T)$

$$T = \{(R_T(\lambda)f, f + \lambda R_T(\lambda)f) : f \in \mathcal{H}\}.$$

We prove (a) $\Rightarrow$ (d) using the observation (compare with [6, 2.2.3]) that (a) is equivalent to  $T$  being a nonpositive subspace of the Krein space  $(\mathcal{H}^2, \langle \cdot, \cdot \rangle_{\mathcal{H}^2})$  with inner product

$$\langle \{f, g\}, \{h, k\} \rangle_{\mathcal{H}^2} := i((g, h)_{\mathcal{H}} - (f, k)_{\mathcal{H}}), \quad \{f, g\}, \{h, k\} \in \mathcal{H}^2.$$

Since (a) is equivalent to (c), we have that  $\mathbb{C}_- \subset \rho(T)$ . Let  $m$  be a positive integer, let  $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{C}_-$  and let  $f_1, f_2, \dots, f_m \in \mathcal{H}$ . Then, in the vector notation explained in the Appendix,

$$-i \left( ((\lambda_j - \lambda_k^*) K_T(\lambda_j, \lambda_k) f_j, f_k)_{\mathcal{H}} \right)_{j,k=1}^m = \langle \mathfrak{G}, \mathfrak{G} \rangle_{\mathcal{H}^2}, \quad (2.1)$$

where  $\mathfrak{G}$  is the row vector  $\mathfrak{G} = (g_1 \ g_2 \ \dots \ g_m)$  with

$$g_j = \{R_T(\lambda_j)f_j, f_j + \lambda_j R_T(\lambda_j)f_j\} \in T, \quad j = 1, 2, \dots, m.$$

By the afore mentioned observation, the matrix on the right hand side of the equality (2.1) is the Gram matrix of  $\mathfrak{G}$  whose entries are elements of the nonpositive subspace  $T$  of  $(\mathcal{H}^2, \langle \cdot, \cdot \rangle_{\mathcal{H}^2})$  and hence it is nonpositive. Thus the matrix

$$i \left( ((\lambda_j - \lambda_k^*) K_T(\lambda_j, \lambda_k) f_j, f_k)_{\mathcal{H}} \right)_{j,k=1}^m$$

is nonnegative. It follows from (6.4) in the Appendix with  $z_j = \lambda_j^*$ ,  $j = 1, 2, \dots, m$ , that the matrix  $(-i/(\lambda_j - \lambda_k^*))_{j,k=1}^m$  is nonnegative. Since the Schur, that is, the entry wise product of two nonnegative matrices is again nonnegative (see [17, p. 9] or [4, Theorem 2.7]), the Schur product

$$((K_T(\lambda_j, \lambda_k) f_j, f_k)_{\mathcal{H}})_{j,k=1}^m = (-i/(\lambda_j - \lambda_k^*))_{j,k=1}^m * i((\lambda_j - \lambda_k^*) K_T(\lambda_j, \lambda_k) f_j, f_k)_{\mathcal{H}})_{j,k=1}^m$$

is nonnegative, that is, the kernel  $K_T(\lambda, \mu)$  is nonnegative on  $\mathbb{C}_-$ . Thus (a) $\Rightarrow$ (d). Evidently (d) $\Rightarrow$ (b).  $\square$

### 3. Compressions of maximal dissipative linear relations

The following theorem and its Krein space version, Theorem 3.3 below, are the main results of this section.

**Theorem 3.1.** *Let  $T$  be a closed dissipative linear relation in a Hilbert space  $\mathcal{H}$ . Let  $T_0$  be the compression of  $T$  to a subspace  $\mathcal{G}$  of  $\mathcal{H}$  with finite codimension:*

$$T_0 = P_{\mathcal{G}} T|_{\mathcal{G}} = \{ \{f, P_{\mathcal{G}} g\} : \{f, g\} \in T, f \in \mathcal{G} \}.$$

Then the following statements are equivalent.

- (i)  $T$  is maximal dissipative in  $\mathcal{H}$ .
- (ii)  $T(0) = T^*(0)$  and  $T_0$  is maximal dissipative in  $\mathcal{G}$ .

The proof of the implication (ii)  $\Rightarrow$  (i) is similar to the proof of [7, Theorem 2.4]. When  $T$  is an operator, the implication (i)  $\Rightarrow$  (ii) in the theorem is due to Nudelman [28] and the reverse implication was shown in [7, Theorem 3.2]. We base the proof of the implication (i)  $\Rightarrow$  (ii) on the next lemma which we formulate in the vector notation discussed in the Appendix; in particular for (3.3) below note (6.5).

**Lemma 3.2.** *Let  $T$  be a linear relation in a Hilbert space  $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$  with  $\rho(T) \neq \emptyset$  and let  $T_0 = P_{\mathcal{G}} T|_{\mathcal{G}}$ , where  $\mathcal{G}$  is a subspace of  $\mathcal{H}$  with  $m := \text{codim } \mathcal{G} < \infty$ . Let  $\mathfrak{B} = (b_1 \ b_2 \ \dots \ b_m)$  be a row vector whose  $m$  entries  $b_j$  form a basis of  $\mathcal{G}^{\perp}$ . Assume that the open set*

$$\rho_d(T) := \{ \lambda \in \rho(T) : \det (R_T(\lambda) \mathfrak{B}, \mathfrak{B})_{\mathcal{H}} \neq 0 \}$$

is nonempty. Then  $\rho_d(T) \subset \rho(T_0)$  and for  $\lambda, \mu \in \rho_d(T)$  and  $g, h \in \mathcal{G}$  the following equalities hold.

$$R_{T_0}(\lambda)g = R_T(\lambda)g - R_T(\lambda)\mathfrak{B} (R_T(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1} (R_T(\lambda)g, \mathfrak{B})_{\mathcal{H}}, \quad (3.1)$$

$$R_{T_0}(\mu)^*g = R_T(\mu)^*g - R_T(\mu)^*\mathfrak{B} (R_T(\mu)^*\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1} (R_T(\mu)^*g, \mathfrak{B})_{\mathcal{H}}, \quad (3.2)$$

$$(K_{T_0}(\lambda, \mu)g, h)_{\mathcal{G}} = x_h(\mu)^* (K_T(\lambda, \mu)(\mathfrak{B} : g), (\mathfrak{B} : h))_{\mathcal{H}} x_g(\lambda), \quad (3.3)$$

where, for example,  $x_g(\lambda)$  is the  $(m+1) \times 1$  vector

$$x_g(\lambda) = \begin{pmatrix} (R_T(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1} (R_T(\lambda)g, \mathfrak{B})_{\mathcal{H}} \\ -1 \end{pmatrix}.$$

The equality (3.1) is taken from [2, Theorem 5.4]. For the convenience of the reader we repeat its proof.

**Proof of Lemma 3.2.** To prove (3.1) we write  $R(\lambda)$  and  $K(\lambda, \mu)$  for  $R_T(\lambda)$  and  $K_T(\lambda, \mu)$  and use that (see (6.3) in the Appendix)

$$P_{\mathcal{G}} = I - \mathfrak{B}(\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(\cdot, \mathfrak{B})_{\mathcal{H}}.$$

Then  $T_0 = \{\{h, P_{\mathcal{G}}k\} : \{h, k\} \in T, h \in \mathcal{G}\}$  implies that

$$\begin{aligned} (T_0 - \lambda)^{-1} &\stackrel{1}{=} \{\{P_{\mathcal{G}}k - \lambda h, h\} : \{h, k\} \in T, h \in \mathcal{G}\} \\ &\stackrel{2}{\subseteq} \{\{g, R(\lambda)(g + \mathfrak{B}(\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(k, \mathfrak{B})_{\mathcal{H}})\} : g = P_{\mathcal{G}}k - \lambda h, \{h, k\} \in T, h \in \mathcal{G}\} \\ &\stackrel{3}{\subseteq} \{\{g, R(\lambda)g - R(\lambda)\mathfrak{B}(R(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(R(\lambda)g, \mathfrak{B})_{\mathcal{H}}\} : g \in \mathcal{G}\} \\ &\stackrel{4}{\subseteq} (T_0 - \lambda)^{-1}. \end{aligned}$$

In deriving the inclusion  $\stackrel{2}{\subseteq}$  we have used that if

$$g := P_{\mathcal{G}}k - \lambda h = k - \lambda h - \mathfrak{B}(\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(k, \mathfrak{B})_{\mathcal{H}}, \quad \{h, k\} \in T,$$

then

$$R(\lambda)g = h - R(\lambda)\mathfrak{B}(\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(k, \mathfrak{B})_{\mathcal{H}}, \quad \{h, k\} \in T.$$

The inclusion  $\stackrel{3}{\subseteq}$  follows from taking the inner product of the elements on both sides of the previous equality with  $\mathfrak{B}$  and using that  $(h, \mathfrak{B})_{\mathcal{H}} = 0$ . We obtain

$$(R(\lambda)g, \mathfrak{B})_{\mathcal{H}} = -(R(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}(\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(k, \mathfrak{B})_{\mathcal{H}}$$

and this implies  $(\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(k, \mathfrak{B})_{\mathcal{H}} = -(R(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(R(\lambda)g, \mathfrak{B})_{\mathcal{H}}$ .

To see the inclusion  $\stackrel{4}{\subseteq}$ , consider  $g \in \mathcal{G}$  and set

$$h := R(\lambda)x, \quad x := g - \mathfrak{B}(R(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(R(\lambda)g, \mathfrak{B})_{\mathcal{H}}, \quad k := x + \lambda h.$$

Then  $\{h, k\} = \{R(\lambda)x, x + \lambda R(\lambda)x\} \in T$ . Moreover,  $(h, \mathfrak{B})_{\mathcal{H}} = (R(\lambda)x, \mathfrak{B})_{\mathcal{H}} = 0$  which implies that  $h \in \mathcal{G}$  and  $g = P_{\mathcal{G}}x = P_{\mathcal{G}}k - \lambda h$ , and hence

$$\{g, R(\lambda)g - R(\lambda)\mathfrak{B}(R(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(R(\lambda)g, \mathfrak{B})_{\mathcal{H}}\} = \{P_{\mathcal{G}}k - \lambda h, h\}.$$

Hence  $\stackrel{4}{\subseteq}$  follows from  $\stackrel{1}{=}$ . We conclude that the inclusions can be replaced by equality signs and hence, identifying an operator with its graph, we see that

$$(T_0 - \lambda)^{-1} = R(\lambda)|_{\mathcal{G}} - R(\lambda)\mathfrak{B}(R(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1}(R(\lambda)|_{\mathcal{G}} \cdot, \mathfrak{B})_{\mathcal{H}}.$$

The operator on the right hand side is bounded and defined on  $\mathcal{G}$ , hence  $(T_0 - \lambda)^{-1}$  is a bounded operator defined on  $\mathcal{G}$ , that is,  $\lambda \in \rho(T_0)$ ,  $(T_0 - \lambda)^{-1} = R_{T_0}(\lambda)$  and  $\rho_d(T) \subset \rho(T_0)$ . This completes the proof of the first part of the lemma.

Equality (3.2) follows in a straightforward manner from (3.1).

Equality (3.3) follows after some calculations from the preceding two. To see this, we write  $R_0(\lambda)$  and  $K_0(\lambda, \mu)$  for  $R_{T_0}(\lambda)$  and  $K_{T_0}(\lambda, \mu)$  and omit the index in the inner product of  $\mathcal{H}$ .

$$\begin{aligned}
 K_0(\lambda, \mu)g &= \frac{R_0(\lambda) - R_0(\mu)^*}{\lambda - \mu^*}g - R_0(\mu)^*R_0(\lambda)g \\
 &\stackrel{(3.2)}{=} \frac{R_0(\lambda) - R_0(\mu)^*}{\lambda - \mu^*}g \\
 &\quad - R(\mu)^*R_0(\lambda)g + R(\mu)^*\mathfrak{B}(R(\mu)^*\mathfrak{B}, \mathfrak{B})^{-1}(R(\mu)^*R_0(\lambda)g, \mathfrak{B}) \\
 &\stackrel{(3.1)}{=} \frac{R_0(\lambda) - R_0(\mu)^*}{\lambda - \mu^*}g \\
 &\quad - R(\mu)^*R(\lambda)g + R(\mu)^*R(\lambda)\mathfrak{B}(R(\lambda)\mathfrak{B}, \mathfrak{B})^{-1}(R(\lambda)g, \mathfrak{B}) \\
 &\quad + R(\mu)^*\mathfrak{B}(R(\mu)^*\mathfrak{B}, \mathfrak{B})^{-1}(R(\mu)^*R(\lambda)g, \mathfrak{B}) \\
 &\quad - R(\mu)^*\mathfrak{B}(R(\mu)^*\mathfrak{B}, \mathfrak{B})^{-1}(R(\mu)^*R(\lambda)\mathfrak{B}, \mathfrak{B})(R(\lambda)\mathfrak{B}, \mathfrak{B})^{-1}(R(\lambda)g, \mathfrak{B}) \\
 &= K(\lambda, \mu)g - K(\lambda, \mu)\mathfrak{B}(R(\lambda)\mathfrak{B}, \mathfrak{B})^{-1}(R(\lambda)g, \mathfrak{B}) \\
 &\quad - R(\mu)^*\mathfrak{B}(R(\mu)^*\mathfrak{B}, \mathfrak{B})^{-1}(K(\lambda, \mu)g, \mathfrak{B}) \\
 &\quad + R(\mu)^*\mathfrak{B}(R(\mu)^*\mathfrak{B}, \mathfrak{B})^{-1}(K(\lambda, \mu)\mathfrak{B}, \mathfrak{B})(R(\lambda)\mathfrak{B}, \mathfrak{B})^{-1}(R(\lambda)g, \mathfrak{B}).
 \end{aligned}$$

The last equality is obtained by replacing  $R(\mu)^*R(\lambda)$  by

$$\frac{R(\lambda) - R(\mu)^*}{\lambda - \mu^*} - K(\lambda, \mu)$$

and observing that the sum of the terms not containing  $K(\lambda, \mu)$  cancel against the quotient  $(R_0(\lambda) - R_0(\mu)^*)/(\lambda - \mu^*)$ . It follows that

$$\begin{aligned}
 (K_0(\lambda, \mu)g, h)_g &= \begin{pmatrix} (R(\mu)\mathfrak{B}, \mathfrak{B})^{-1}(R(\mu)h, \mathfrak{B}) \\ -1 \end{pmatrix}^* \\
 &\quad \times \begin{pmatrix} (K(\lambda, \mu)\mathfrak{B}, \mathfrak{B}) & (K(\lambda, \mu)g, \mathfrak{B}) \\ (K(\lambda, \mu)\mathfrak{B}, h) & (K(\lambda, \mu)g, h) \end{pmatrix} \begin{pmatrix} (R(\lambda)\mathfrak{B}, \mathfrak{B})^{-1}(R(\lambda)g, \mathfrak{B}) \\ -1 \end{pmatrix},
 \end{aligned}$$

which is the expanded form of the equality (3.3).  $\square$

**Proof of Theorem 3.1.** (i)  $\Rightarrow$  (ii): The equality in (ii) follows from Lemma 2.1 and the remark following it. To prove the second part of (ii) we first assume that

$$(\mathcal{H} \ominus \mathcal{G}) \cap T(0) = \{0\}. \quad (3.4)$$

By Lemma 2.2, we have that  $\mathbb{C}_- \subset \rho(T)$  and

$$\lim_{y \in \mathbb{R}, y \rightarrow -\infty} -iy(R_T(iy)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}} = (P_0\mathfrak{B}, \mathfrak{B})_{\mathcal{H}},$$

where  $P_0$  is the orthogonal projection in  $\mathcal{H}$  onto  $T(0)^\perp$ . By (3.4), the  $m \times m$  matrix on the right hand side is invertible, hence  $\rho_d(T) \cap \mathbb{C}_- \neq \emptyset$ . Choose  $\lambda \in \rho_d(T) \cap \mathbb{C}_-$ . Then, since  $T$  is maximal dissipative and by Lemma 2.3,  $K_T(\lambda, \lambda) \geq 0$ . Lemma 3.2 and the identity (3.3) imply that  $\lambda \in \rho(T_0)$  and  $K_{T_0}(\lambda, \lambda) \geq 0$ . Hence, again according to Lemma 2.3,  $T_0$  is maximal dissipative.



Now we drop the assumption (3.4) and define the subspaces

$$\mathcal{F}_0 = (\mathcal{H} \ominus \mathcal{G}) \cap T(0), \quad \mathcal{F}_1 = (\mathcal{H} \ominus \mathcal{G}) \ominus \mathcal{F}_0.$$

Then

$$\mathcal{H} = \mathcal{G} \oplus \mathcal{F}_0 \oplus \mathcal{F}_1 = T(0)^\perp \oplus (T(0) \ominus \mathcal{F}_0) \oplus \mathcal{F}_0.$$

Set

$$\mathcal{H}_1 = \mathcal{G} \oplus \mathcal{F}_1$$

and denote by  $P_1$  the orthogonal projection in  $\mathcal{H}$  onto  $\mathcal{H}_1$ . Then, using the representation  $T = T_{\text{op}} \dot{+} T_\infty$  as in Lemma 2.2 (4), we get that

$$T_1 := P_1 T|_{\text{ran } P_1} = T_{\text{op}} \dot{+} (\{0\} \times T_1(0)), \quad T_1(0) = P_1 T(0) = T(0) \ominus \mathcal{F}_0.$$

It follows from Lemma 2.2 that  $T_1$  is a maximal dissipative linear relation in  $\mathcal{H}_1$ . Let  $P_2$  be the orthogonal projection in  $\mathcal{H}_1$  onto  $\mathcal{G}$ . Then

$$T_0 = P_{\mathcal{G}} T|_{\mathcal{G}} = P_2 T_1|_{\text{ran } P_2}$$

and

$$(\mathcal{H}_1 \ominus \text{ran } P_2) \cap T_1(0) = \mathcal{F}_1 \cap (T(0) \ominus \mathcal{F}_0) = \{0\}.$$

The last equality is the analog of (3.4) for this case. Thus, by what has been shown in the first part of this proof,  $T_0$  is maximal dissipative in  $\mathcal{G}$ .

(ii)  $\Rightarrow$  (i): Choose a point  $\lambda \in \mathbb{C}_- \subset \rho(T_0)$  (see Lemma 2.2). Then, since  $T$  is dissipative,  $(T - \lambda)^{-1}$  is a well defined operator from  $\text{ran } (T - \lambda)$  to  $\text{dom } T$ . Let  $T_{\text{res}}$  be the restriction of  $T$  to  $\mathcal{G}$ :  $T_{\text{res}} = T|_{\mathcal{G}} := T \cap (\mathcal{G} \times \mathcal{H})$ . Then  $P_{\mathcal{G}} T_{\text{res}} = T_0$  and  $T_{\text{res}}$  is closed, because  $T$  is closed. We claim that  $\text{ran } (T_{\text{res}} - \lambda)$  is closed and that  $P_{\mathcal{G}}|_{\text{ran } (T_{\text{res}} - \lambda)}$  is a surjective mapping from  $\text{ran } (T_{\text{res}} - \lambda)$  onto  $\mathcal{G}$  with kernel  $\mathcal{G}^\perp \cap T(0)$ . The claim will be proved later, for now we assume that it is true. Then

$$\text{codim } \text{ran } (T_{\text{res}} - \lambda) = \dim \left( \mathcal{G}^\perp \ominus (\mathcal{G}^\perp \cap T(0)) \right) < \infty \quad (3.5)$$

and, since  $\text{ran } (T_{\text{res}} - \lambda) \subset \text{ran } (T - \lambda)$ , the range  $\text{ran } (T - \lambda)$  is closed and

$$\text{ran } (T - \lambda) = D + \text{ran } (T_{\text{res}} - \lambda), \quad \text{direct sum}, \quad (3.6)$$

for some linear subset  $D$  of  $\mathcal{H}$  with

$$\dim D \leq \dim \left( \mathcal{G}^\perp \ominus (\mathcal{G}^\perp \cap T(0)) \right). \quad (3.7)$$

Applying  $(T - \lambda)^{-1}$  to both sides of (3.6) we obtain

$$\text{dom } T = (T - \lambda)^{-1} D + \text{dom } T_{\text{res}} \subset (T - \lambda)^{-1} D + \mathcal{G}. \quad (3.8)$$

We assume that in (3.7) the equality is strict and derive a contradiction. The assumption implies that there is a nonzero element  $x \in \mathcal{G}^\perp \ominus (\mathcal{G}^\perp \cap T(0))$  such that  $((T - \lambda)^{-1} D, x)_{\mathcal{H}} = \{0\}$ . Then, by (3.8),  $(\text{dom } T, x)_{\mathcal{H}} = \{0\}$ . Since, by the assumption that  $T^*(0) = T(0)$ ,  $\text{dom } T$  is dense in  $T(0)^\perp$ , we conclude that  $x \in T(0)$ , that is,

$$0 \neq x \in (\mathcal{G}^\perp \cap T(0)) \cap \left( \mathcal{G}^\perp \ominus (\mathcal{G}^\perp \cap T(0)) \right) = \{0\}.$$

This contradiction implies that in (3.7) equality prevails and hence, on account of (3.5) and (3.6),  $\text{ran}(T - \lambda) = \mathcal{H}$ , that is,  $\lambda \in \rho(T)$ . Lemma 2.2 implies that  $T$  is maximal dissipative.

It remains to prove the claim. Let  $y \in \overline{\text{ran}(T_{\text{res}} - \lambda)}$  and let  $y_n \in \text{ran}(T_{\text{res}} - \lambda)$  be a sequence which converges to  $y$ . Then there are  $x_n \in \mathcal{G}$  such that  $\{x_n, y_n + \lambda x_n\} \in T_{\text{res}} \Rightarrow \{x_n, P_{\mathcal{G}}y_n + \lambda x_n\} \in P_{\mathcal{G}}T_{\text{res}} = T_0 \Rightarrow \{x_n, P_{\mathcal{G}}y_n\} \in T_0 - \lambda \Rightarrow x_n = (T_0 - \lambda)^{-1}P_{\mathcal{G}}y_n$ . Since  $(T_0 - \lambda)^{-1}P_{\mathcal{G}}$  is bounded, the sequence  $x_n$  converges to  $x := (T_0 - \lambda)^{-1}P_{\mathcal{G}}y \in \mathcal{G}$ . Thus  $\{x, y + \lambda x\} \in T_{\text{res}}$ , that is,  $y \in \text{ran}(T_{\text{res}} - \lambda)$ . This proves that the range  $\text{ran}(T_{\text{res}} - \lambda)$  is closed. From  $P_{\mathcal{G}}\text{ran}(T_{\text{res}} - \lambda) = \text{ran}(T_0 - \lambda) = \mathcal{G}$  it follows that the operator  $P_{\mathcal{G}}|_{\text{ran}(T_{\text{res}} - \lambda)}$  is surjective. Let  $y \in \text{ran}(T_{\text{res}} - \lambda)$  belong to the kernel of this operator. Then there is an  $x \in \mathcal{G}$  such that  $\{x, y + \lambda x\} \in T_{\text{res}}$  and hence  $\{x, \lambda x\} \in P_{\mathcal{G}}T_{\text{res}} = T_0$ . Since  $T_0$  is dissipative and  $\text{Im } \lambda < 0$  we have that  $x = 0$ . Thus  $y \in T_{\text{res}}(0) = T(0)$ . We have shown that  $\mathcal{G}^{\perp} \cap \text{ran}(T_{\text{res}} - \lambda) \subset \mathcal{G}^{\perp} \cap T(0)$ . In this inclusion equality prevails, because  $\mathcal{G}^{\perp} \cap T(0) \subset \text{ran}(T_{\text{res}} - \lambda)$ . This completes the proof of the claim.  $\square$

The following theorem is a generalization of [7, Theorem 3.4] to linear relations.

**Theorem 3.3.** *Let  $T$  be a closed dissipative linear relation in a Krein space  $\mathcal{K}$ . Let  $T_0$  be the compression of  $T$  to a Krein subspace  $\mathcal{G}$  of  $\mathcal{K}$  with finite codimension. Then  $T$  is maximal dissipative in  $\mathcal{K}$  if and only if  $T(0) = T^*(0)$  and  $T_0$  is maximal dissipative in  $\mathcal{G}$ .*

**Proof.** Let  $J$  be a fundamental symmetry on  $\mathcal{K}$  whose restriction to  $\mathcal{G}$  is also a fundamental symmetry. Denote by  $\mathcal{K}_J$  the Hilbert space with inner product  $(x, y)_J := (Jx, y)_{\mathcal{K}}$ . Then  $P_{\mathcal{G}} = JP_1J$  where  $P_1$  is the orthogonal projection in  $\mathcal{K}_J$  onto  $\mathcal{G}$  and  $JT_0 = P_1JT|_{\text{ran } P_1}$ . The theorem now follows from Theorem 3.1 and the fact that  $T(T_0)$  is maximal dissipative in  $\mathcal{K}(\mathcal{G})$  if and only if  $JT(JT_0)$  is maximal dissipative in the Hilbert space  $\mathcal{K}_J(\mathcal{G}_J)$  (see [6, 2.2.3] for the operator case).  $\square$

**Corollary 3.4.** *Let  $T$  be a closed linear relation in a Krein space  $\mathcal{K}$ . Let  $\mathbb{P}$  be a set of orthogonal projections in  $\mathcal{K}$  whose ranges are regular subspaces of  $\mathcal{K}$  with finite codimension and cover  $\mathcal{K}$ :*

$$\mathcal{K} = \bigcup_{P \in \mathbb{P}} \text{ran } P.$$

- (a) *If  $PT|_{\text{ran } P}$  is dissipative in  $\text{ran } P$  for each  $P \in \mathbb{P}$ , then  $T$  is dissipative in  $\mathcal{K}$ .*
- (b) *If  $PT|_{\text{ran } P}$  is dissipative in  $\text{ran } P$  for each  $P \in \mathbb{P}$  and maximal dissipative for at least one  $P \in \mathbb{P}$  and if  $T(0) = T^*(0)$ , then  $T$  is maximal dissipative in  $\mathcal{K}$ .*

**Proof.** (a) We assume that  $T$  is not dissipative and derive a contradiction. The assumption implies there is an element  $\{f, g\} \in T$  such that  $\text{Im}(g, f)_{\mathcal{K}} < 0$ . Choose  $P \in \mathbb{P}$  such that  $f \in \text{ran } P$ . Then  $\{f, Pg\} \in PT|_{\text{ran } P}$  and hence, since  $PT|_{\text{ran } P}$  is dissipative,

$$0 \leq \text{Im}(Pg, f)_{\mathcal{K}} = \text{Im}(g, f)_{\mathcal{K}} < 0.$$

This contradiction shows that  $T$  is dissipative.

(b) By (a), the first assumption in (b) implies that  $T$  is dissipative. The conclusion then follows from Theorem 3.3.  $\square$

#### 4. Compressions of self-adjoint linear relations

The following theorem is a generalization of [7, Theorem 3.4] to linear relations. If  $\mathcal{K}$  is a Hilbert space and  $S$  is an operator in  $\mathcal{K}$ , then the only if part is due to Stenger [30]. If  $\mathcal{K}$  is a Pontryagin space and  $S$  is a self-adjoint linear relation in  $\mathcal{K}$  with  $\rho(S) \neq \emptyset$ , then the only if part is proved in [13, Theorem 3.3] and the remark following it; this result was obtained in connection with Straus extensions of a symmetric linear relation.

**Theorem 4.1.** Let  $S$  be a closed symmetric linear relation in a Krein space  $\mathcal{K}$ . Let  $S_0$  be the compression of  $S$  to a Krein subspace  $\mathcal{G}$  of  $\mathcal{K}$  with finite codimension:  $S_0 = P_{\mathcal{G}}S|_{\mathcal{G}}$ . Then  $S$  is self-adjoint in  $\mathcal{K}$  if and only if  $S(0) = S^*(0)$  and  $S_0$  is self-adjoint in  $\mathcal{G}$ .

A proof of Theorem 4.1 can be based on Lemma 3.2 by using that in a Hilbert space a linear relation  $A$  is self-adjoint if and only if there is a  $v \in \mathbb{C} \setminus \mathbb{R}$ , such that  $v, v^* \in \rho(A)$  and

$$R(\lambda) - R(\mu)^* = (\lambda - \mu^*)R(\mu)^*R(\lambda), \quad \lambda, \mu \in \{v, v^*\},$$

(see for example [15, Theorem 4.1] and its proof) and by adapting the lemmas and the theorems of Section 2 and Section 3 and their proofs to the self-adjoint case. More simply, Theorem 4.1 can also be seen as a corollary of Theorem 3.3 by observing that if a linear relation in  $\mathcal{K}$  is symmetric, then it is dissipative, and that a linear relation  $S$  is self-adjoint if and only if  $S$  and  $-S$  are maximal dissipative.

The proof of the following corollary is similar to the proof of Corollary 3.4.

**Corollary 4.2.** Let  $S$  be a closed linear relation in a Krein space  $\mathcal{K}$ . Let  $\mathbb{P}$  be a set of orthogonal projections in  $\mathcal{K}$  whose ranges are regular subspaces of  $\mathcal{K}$  with finite codimension and cover  $\mathcal{K}$ :

$$\mathcal{K} = \bigcup_{P \in \mathbb{P}} \text{ran } P.$$

(a) If  $PS|_{\text{ran } P}$  is symmetric in  $\text{ran } P$  for each  $P \in \mathbb{P}$ , then  $S$  is symmetric in  $\mathcal{K}$ .

(b) If  $PS|_{\text{ran } P}$  is symmetric in  $\text{ran } P$  for each  $P \in \mathbb{P}$  and self-adjoint for at least one  $P \in \mathbb{P}$  and  $S(0) = S^*(0)$ , then  $S$  is self-adjoint in  $\mathcal{K}$ .

In the remainder of this section  $S$  is a symmetric linear relation in a Hilbert space  $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$  which is nonnegative, which means that

$$(g, f)_{\mathcal{H}} \geq 0, \quad \{f, g\} \in S.$$

Then there are two special nonnegative self-adjoint extensions  $S_{\mu}$  and  $S_M$  of  $S$  with the property that a nonnegative self-adjoint linear relation  $H$  in  $\mathcal{H}$  is an extension of  $S$  (that is,  $S \subset H$ ) if and only if  $S_M \leq H \leq S_{\mu}$  in the sense that

$$(S_{\mu} + \alpha)^{-1} \leq (H + \alpha)^{-1} \leq (S_M + \alpha)^{-1} \quad \text{for all } \alpha > 0.$$

The linear relation  $S_{\mu}$  is called the hard or Friedrichs extension of  $S$  and is given by

$$S_{\mu} = \{\{f, g\} \in S^* : \exists \{f_n, g_n\} \in S \text{ such that } f_n \rightarrow f, (g_n - g_m, f_n - f_m)_{\mathcal{H}} \rightarrow 0\},$$

the linear relation  $S_M$  is called the soft extension of  $S$  and is given by  $S_M = ((S^{-1})_{\mu})^{-1}$ , that is,

$$S_M = \{\{f, g\} \in S^* : \exists \{f_n, g_n\} \in S \text{ such that } g_n \rightarrow g, (f_n - f_m, g_n - g_m)_{\mathcal{H}} \rightarrow 0\}.$$

It follows from the formulas for  $S_{\mu}$  and  $S_M$  that

$$(i) \quad S_{\mu}(0) = S^*(0) \quad \text{and} \quad (ii) \quad \ker S_M = \ker S^*. \quad (4.1)$$

Since  $\overline{\text{dom } S} = S^*(0)^{\perp}$ , equality (i) implies that  $S_{\mu}$  is an operator if and only if  $S$  is densely defined (and hence an operator). In that case all self-adjoint extensions of  $S$  are operators including  $S_M$ . These

extreme extensions of nonnegative symmetric linear relations were defined and studied by Coddington and de Snoo [11]. There  $S_\mu$  and  $S_M$  are denoted by  $S_F$  and  $S_N$ ; here we follow the notation and terminology of Krein [23].

In what follows we address the following two questions.

**Problem 4.3.** Let  $S$  be a nonnegative symmetric linear relation in a Hilbert space  $\mathcal{H}$  and let  $\mathcal{G}$  be a subspace of  $\mathcal{H}$  with finite co-dimension. When is the compression of the hard and the soft extension of  $S$  to  $\mathcal{G}$  equal to the hard and the soft extension of the compression of  $S$  to  $\mathcal{G}$ , in formula: When is

$$(i) \quad P_{\mathcal{G}}S_\mu|_{\mathcal{G}} = (P_{\mathcal{G}}S|_{\mathcal{G}})_\mu \quad \text{and} \quad (ii) \quad P_{\mathcal{G}}S_M|_{\mathcal{G}} = (P_{\mathcal{G}}S|_{\mathcal{G}})_M? \quad (4.2)$$

We first consider the hard and then the soft extension. For a closed symmetric relation  $S$  in a Hilbert space  $\mathcal{H}$  and a subspace  $\mathcal{G}$  of  $\mathcal{H}$  with finite codimension we list the following statements which will serve as conditions in the main theorems.

$$\begin{aligned} (1) \quad & S \text{ is densely defined.} \quad (2) \quad S^*(0) \cap \mathcal{G}^\perp = \{0\}. \\ (3) \quad & P_{\mathcal{G}}S^* \text{ is closed.} \quad (4) \quad (P_{\mathcal{G}}S|_{\mathcal{G}})^* = P_{\mathcal{G}}S^*|_{\mathcal{G}}. \end{aligned} \quad (4.3)$$

The adjoint on the left hand side of the equality (4) is taken in  $\mathcal{G}$  (not in  $\mathcal{H}$  as is done on the right hand side).

**Lemma 4.4.** We have  $(1) \Rightarrow (2)$ ,  $(2) \Rightarrow (3)$  and  $(3) \Rightarrow (4)$ .

**Proof.** The first implication holds since  $\overline{\text{dom } S} = S^*(0)^\perp$ .

We prove the second implication. Let  $\{x_n, y_n\}$  be a sequence in  $P_{\mathcal{G}}S^*$  and assume  $\{x_n, y_n\} \rightarrow \{x, y\}$ . Then there exist elements  $z_n \in \mathcal{H}$  such that  $\{x_n, z_n\} \in S^*$  and  $P_{\mathcal{G}}z_n = y_n$ . Write  $z_n$  as  $z_n = y_n + h_n$  with  $y_n \in \mathcal{G}$  and  $h_n \in \mathcal{G}^\perp$ . Denote the norm in  $\mathcal{H}$  by  $\|\cdot\|_{\mathcal{H}}$ . We assume  $\|h_n\|_{\mathcal{H}} \rightarrow \infty$  and derive a contradiction. Since  $\mathcal{G}^\perp$  is finite dimensional, the assumption implies that  $h_n/\|h_n\|_{\mathcal{H}}$  (or a subsequence which we identify with the sequence)  $\rightarrow h$  for some  $h \in \mathcal{G}^\perp$  with  $\|h\|_{\mathcal{H}} = 1$ . Then, because  $x_n/\|h_n\|_{\mathcal{H}} \rightarrow 0$  and  $y_n/\|h_n\|_{\mathcal{H}} \rightarrow 0$ , we have  $\{x_n/\|h_n\|_{\mathcal{H}}, z_n/\|h_n\|_{\mathcal{H}}\} \rightarrow \{0, h\}$ . Since  $\{x_n/\|h_n\|_{\mathcal{H}}, z_n/\|h_n\|_{\mathcal{H}}\} \in S^*$  and  $S^*$  is closed, we have that  $\{0, h\} \in S^*$ , that is,  $h \in S^*(0) \cap \mathcal{G}^\perp = \{0\}$ . Hence  $0 = \|h\|_{\mathcal{H}} = 1$ . This contradiction implies that we may assume  $h_n$  (or a subsequence) bounded. Then, because  $\mathcal{G}^\perp$  is finite dimensional,  $h_n$  (or a subsequence) converges to some  $h \in \mathcal{G}^\perp$ . Then  $z_n = y_n + h_n \rightarrow y + h$  and  $\{x, y + h\} \in S^*$ . This implies that  $\{x, y\} \in P_{\mathcal{G}}S^*$ . Hence  $P_{\mathcal{G}}S^*$  is closed.

We now prove the third implication. For that we use that if  $A$  is a bounded operator and  $B$  is a linear relation on  $\mathcal{H}$ , then  $(AB)^* = B^*A^*$ . If we apply this twice and use that  $S^{**} = S$  and  $P_{\mathcal{G}}^* = P_{\mathcal{G}}$  we obtain that

$$(P_{\mathcal{G}}SP_{\mathcal{G}})^* = (P_{\mathcal{G}}(P_{\mathcal{G}}S^*)^*)^* = (P_{\mathcal{G}}S^*)^{**}P_{\mathcal{G}} = P_{\mathcal{G}}S^*P_{\mathcal{G}}.$$

Restricting this formula to  $\mathcal{G}$  we readily obtain the equality  $(P_{\mathcal{G}}S|_{\mathcal{G}})^* = P_{\mathcal{G}}S^*|_{\mathcal{G}}$ .  $\square$

The following theorem shows that (4.2) (i) holds if  $S$  is densely defined and hence an operator; the weaker assumptions (2)–(4) allow  $S$  to be a linear relation.

**Theorem 4.5.** Let  $S$  be a closed nonnegative symmetric linear relation in a Hilbert space  $\mathcal{H}$  and let  $\mathcal{G}$  be a subspace of  $\mathcal{H}$  with finite codimension. Assume at least one of the statements (1)–(4) in (4.3) holds. Then  $P_{\mathcal{G}}S_\mu|_{\mathcal{G}} = (P_{\mathcal{G}}S|_{\mathcal{G}})_\mu$ .

As a simple example consider the case where  $S = P_{\mathcal{G}}S|_{\mathcal{G}}$ . Then  $S^* = (P_{\mathcal{G}}S|_{\mathcal{G}})^* \dot{+} (\mathcal{G}^\perp \times \mathcal{G}^\perp)$ , where the adjoint of  $P_{\mathcal{G}}S|_{\mathcal{G}}$  is taken in  $\mathcal{G}$ . It follows that  $\mathcal{G}^\perp \subset S^*(0)$  but  $P_{\mathcal{G}}S^* = (P_{\mathcal{G}}S|_{\mathcal{G}})^* \dot{+} (\mathcal{G}^\perp \times \{0\})$  is closed. Thus (4.3) (3) is valid and the theorem implies that, in this case, the equality (4.2) (i) holds.

**Proof of Theorem 4.5.** According to Lemma 4.4 we may assume that (4.3) (4) holds. By Theorem 4.1, the linear relations  $P_{\mathcal{G}}S_{\mu}|_{\mathcal{G}}$  and  $(P_{\mathcal{G}}S|_{\mathcal{G}})_{\mu}$  are self-adjoint in  $\mathcal{G}$ , and therefore to show that they are equal it suffices to show that

$$(P_{\mathcal{G}}S|_{\mathcal{G}})_{\mu} \subset P_{\mathcal{G}}S_{\mu}|_{\mathcal{G}}. \quad (4.4)$$

Let  $\{f, g\} \in (P_{\mathcal{G}}S|_{\mathcal{G}})_{\mu}$ . Then  $\{f, g\} \in (P_{\mathcal{G}}S|_{\mathcal{G}})^* = P_{\mathcal{G}}S^*|_{\mathcal{G}}$  and

$$\exists \{f_n, g_n\} \in P_{\mathcal{G}}S|_{\mathcal{G}} : \lim_{n \rightarrow \infty} f_n = f, \lim_{n, m \rightarrow \infty} (f_n - f_m, g_n - g_m)_{\mathcal{H}} = 0.$$

This implies that for some  $h, h_n \in \mathcal{H}$  with  $P_{\mathcal{H}}h = g$  and  $P_{\mathcal{G}}h_n = g_n$  we have  $\{f, h\} \in S^*$  and

$$\{f_n, h_n\} \in S, \lim_{n \rightarrow \infty} f_n = f, \lim_{n, m \rightarrow \infty} (f_n - f_m, h_n - h_m)_{\mathcal{H}} = 0.$$

Here we have used that  $f_n \in \mathcal{G}$ . It follows that  $\{f, h\} \in S_{\mu}$ . Since also  $f \in \mathcal{G}$  we see that  $\{f, g\} = \{f, P_{\mathcal{G}}h\} \in P_{\mathcal{G}}S_{\mu}|_{\mathcal{G}}$ . This implies (4.4).

Even for a closed densely defined nonnegative symmetric operator  $S$  the equality (4.2) (ii) need not be true as the following example shows.

**Example 4.6.** Let  $S$  be a closed densely defined nonnegative symmetric operator in a Hilbert space  $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$  which is not self-adjoint:  $S \neq S^*$ . Choose elements  $x_0 \in \ker(S^* - i)$  and  $y_0 \in \ker(S^* + i)$  with norms  $\|x_0\|_{\mathcal{H}} = \|y_0\|_{\mathcal{H}} = 1$  such that the inner product  $(x_0, y_0)_{\mathcal{H}}$  is real. Then  $f_0 := x_0 - y_0 \in \operatorname{dom} S^* \setminus \ker S^*$

$$\begin{aligned} (S^*f_0, f_0)_{\mathcal{H}} &= i(x_0 + y_0, x_0 - y_0)_{\mathcal{H}} \\ &= i(x_0, x_0)_{\mathcal{H}} + i(y_0, x_0)_{\mathcal{H}} - i(y_0, x_0)_{\mathcal{H}} - i(y_0, y_0)_{\mathcal{H}} = 0. \end{aligned}$$

Let  $\mathcal{G} = \{S^*f_0\}^{\perp}$ . Then

$$\ker P_{\mathcal{G}}S_M|_{\mathcal{G}} = \ker S^* \cap \mathcal{G}. \quad (4.5)$$

To see this we first note that, by (4.1) (ii), the set on the right hand side of (4.5) is a subset of the set on the left hand side. To prove the converse inclusion consider  $g \in \ker P_{\mathcal{G}}S_M|_{\mathcal{G}}$ . Then  $g \in \mathcal{G} \cap \operatorname{dom} S_M$ ,  $S_M g \in \mathcal{G}^{\perp}$  and hence  $(S_M g, g)_{\mathcal{H}} = 0$ . It follows that  $S_M^{1/2} g = 0$ , hence  $S^*g = S_M g = 0$ , which implies  $g \in \ker S^* \cap \mathcal{G}$ . This completes the proof of the equality (4.5). From (4.1) (ii) and Lemma 4.4 it follows that

$$\ker(P_{\mathcal{G}}S|_{\mathcal{G}})_M = \ker(P_{\mathcal{G}}S|_{\mathcal{G}})^* = \ker P_{\mathcal{G}}S^*|_{\mathcal{G}}. \quad (4.6)$$

The equality (4.5) and  $f_0 \notin \ker S^*$  imply that  $f_0 \notin \ker P_{\mathcal{G}}S_M|_{\mathcal{G}}$  and the equality (4.6),  $f_0 \in \mathcal{G}$  and  $S^*f_0 \in \mathcal{G}^{\perp}$  imply that  $f_0 \in \ker(P_{\mathcal{G}}S|_{\mathcal{G}})_M$ . It follows that the kernels  $\ker P_{\mathcal{G}}S_M|_{\mathcal{G}}$  and  $\ker(P_{\mathcal{G}}S|_{\mathcal{G}})_M$  do not coincide and therefore

$$P_{\mathcal{G}}S_M|_{\mathcal{G}} \neq (P_{\mathcal{G}}S|_{\mathcal{G}})_M.$$

**Theorem 4.7.** Let  $S$  be a closed nonnegative symmetric relation in a Hilbert space  $\mathcal{H}$  and let  $\mathcal{G}$  be a subspace of  $\mathcal{H}$  with finite codimension. Assume that at least one of the statements (1)–(4) in (4.3) is valid. Then

$$P_{\mathcal{G}}S_M|_{\mathcal{G}} = (P_{\mathcal{G}}S|_{\mathcal{G}})_M \Rightarrow \operatorname{ran} S^*|_{\mathcal{G}} \cap \mathcal{G}^{\perp} = \{0\} \quad (4.7)$$

and

$$\overline{\text{ran } S^*|_{\mathcal{G}}} \cap \mathcal{G}^\perp = \{0\} \Rightarrow P_{\mathcal{G}} S_M|_{\mathcal{G}} = (P_{\mathcal{G}} S|_{\mathcal{G}})_M. \quad (4.8)$$

**Proof.** By Lemma 4.4, we may assume that (4.3) (4) holds. We prove the implication (4.7). As in Example 4.6 one can show that

$$\ker P_{\mathcal{G}} S_M|_{\mathcal{G}} = \ker S^* \cap \mathcal{G} = \ker S^*|_{\mathcal{G}}$$

and, by (4.1) (ii) and (4.3) (4), we have

$$\ker(P_{\mathcal{G}} S|_{\mathcal{G}})_M = \ker(P_{\mathcal{G}} S|_{\mathcal{G}})^* = \ker P_{\mathcal{G}} S^*|_{\mathcal{G}}.$$

The assumption  $P_{\mathcal{G}} S_M|_{\mathcal{G}} = (P_{\mathcal{G}} S|_{\mathcal{G}})_M$  implies  $\ker P_{\mathcal{G}} S^*|_{\mathcal{G}} = \ker S^*|_{\mathcal{G}}$ , which is equivalent to  $\text{ran } S^*|_{\mathcal{G}} \cap \mathcal{G}^\perp = \{0\}$ . This proves (4.7).

We now prove the implication (4.8). The proof is similar to the proof of Theorem 4.5. By Theorem 4.1, the linear relations  $P_{\mathcal{G}} S_M|_{\mathcal{G}}$  and  $(P_{\mathcal{G}} S|_{\mathcal{G}})_M$  are self-adjoint in  $\mathcal{G}$ , and therefore to show that they are equal it suffices to show that

$$(P_{\mathcal{G}} S|_{\mathcal{G}})_M \subset P_{\mathcal{G}} S_M|_{\mathcal{G}}. \quad (4.9)$$

Let  $\{f, g\} \in (P_{\mathcal{G}} S|_{\mathcal{G}})_M$ , then  $\{f, g\} \in (P_{\mathcal{G}} S|_{\mathcal{G}})^* = P_{\mathcal{G}} S^*|_{\mathcal{G}}$  and

$$\exists \{f_n, g_n\} \in P_{\mathcal{G}} S|_{\mathcal{G}} : \lim_{n \rightarrow \infty} g_n = g, \quad \lim_{n, m \rightarrow \infty} (f_n - f_m, g_n - g_m)_{\mathcal{H}} = 0.$$

This implies that for some  $h, h_n \in \mathcal{H}$  with  $P_{\mathcal{H}} h = g$  and  $P_{\mathcal{G}} h_n = g_n$  we have  $\{f, h\} \in S^*$  and, since  $f_n \in \mathcal{G}$ ,

$$\{f_n, h_n\} \in S, \quad \lim_{n, m \rightarrow \infty} (f_n - f_m, h_n - h_m)_{\mathcal{H}} = 0.$$

We claim that also  $\lim_{n \rightarrow \infty} h_n = h$ . Assume the claim is correct. Then  $\{f, h\} \in S_M$  and, since  $f \in \mathcal{G}$ , we see that  $\{f, g\} = \{f, P_{\mathcal{G}} h\} \in P_{\mathcal{G}} S_M|_{\mathcal{G}}$ . This implies the inclusion (4.9) and the proof of the implication (4.8) is complete. It remains to prove the claim. For this we use the assumption  $\overline{\text{ran } S^*|_{\mathcal{G}}} \cap \mathcal{G}^\perp = \{0\}$ . Rewriting this equality in the form

$$((\text{ran } S^*|_{\mathcal{G}})^\perp + \mathcal{G})^\perp = \{0\},$$

we see that it is equivalent to the equality

$$\overline{(\text{ran } S^*|_{\mathcal{G}})^\perp + \mathcal{G}} = \mathcal{H},$$

and hence Lemma 6.1 (ii) below with  $\mathcal{F} \subset \mathcal{D} := (\text{ran } S^*|_{\mathcal{G}})^\perp + \mathcal{G}$  implies

$$\mathcal{H} = \mathcal{F} + \mathcal{G} \subset (\text{ran } S^*|_{\mathcal{G}})^\perp + \mathcal{G} \subset \mathcal{H},$$

that is,

$$(\text{ran } S^*|_{\mathcal{G}})^\perp + \mathcal{G} = \mathcal{H}. \quad (4.10)$$

Consider  $k \in \mathcal{H}$  and write it as  $k = k_1 + k_2$  according to the decomposition (4.10), that is, with  $k_1 \in (\text{ran } S^*|_{\mathcal{G}})^{\perp}$  and  $k_2 \in \mathcal{G}$ . Then, because  $h \in \text{ran } S^*|_{\mathcal{G}}$  and  $h_n \in \text{ran } S|_{\mathcal{G}} \subset \text{ran } S^*|_{\mathcal{G}}$ ,

$$(h_n - h, k)_{\mathcal{H}} = (h_n - h, k_2)_{\mathcal{H}} = (P_{\mathcal{G}}h_n - P_{\mathcal{G}}h, k_2)_{\mathcal{H}} = (g_n - g, k_2)_{\mathcal{H}} \rightarrow 0.$$

This shows that  $h_n - h$  converges weakly in  $\mathcal{H}$  to 0. Since the operator  $I - P_{\mathcal{G}}$  is compact (has finite dimensional range) it follows that  $(I - P_{\mathcal{G}})(h_n - h) \rightarrow 0$ . This together with  $P_{\mathcal{G}}(h_n - h) = g_n - g \rightarrow 0$  implies that  $h_n \rightarrow h$ .  $\square$

For proofs of the following lemma see [22, Lemma 324], [25, Theorem 5.1] or [18, Theorem IV.1.2].

**Lemma 4.8.** *If  $T$  is a closed densely defined operator on a Hilbert space, then  $\text{ran } T$  is closed if and only if  $\text{ran } T^*$  is closed.*

**Corollary 4.9.** *Let  $S$  be a closed densely defined nonnegative symmetric operator in a Hilbert space  $\mathcal{H}$  and let  $\mathcal{G}$  be a subspace of  $\mathcal{H}$  with finite codimension. Assume that  $P_{\mathcal{G}}\text{ran } S$  is closed. Then*

$$P_{\mathcal{G}}S_M|_{\mathcal{G}} = (P_{\mathcal{G}}S|_{\mathcal{G}})_M \Leftrightarrow \text{ran } S^*|_{\mathcal{G}} \cap \mathcal{G}^{\perp} = \{0\}.$$

Since  $\text{ran } P_{\mathcal{G}}S = P_{\mathcal{G}}\text{ran } S$  is closed, Lemma 4.8 applied to  $T = P_{\mathcal{G}}S$  implies  $\text{ran } S^*|_{\mathcal{G}}$  is also closed. Hence the corollary follows from Theorem 4.7.

**Remark 4.10.** Krein [23] (see also [11, Theorem 7]) shows that associated with a bounded symmetric operator  $A$  in a Hilbert space  $\mathcal{H}$  with norm  $\|A\|_{\mathcal{H}} \leq 1$  there are self-adjoint contractions  $A_{\mu}$  and  $A_M$  on  $\mathcal{H}$  which are extensions of  $A$  with the property that for every self-adjoint contraction  $H$  on  $\mathcal{H}$  the following two statements are equivalent:

$$(a) \quad A \subset H \qquad (b) \quad A_{\mu} \leq H \leq A_M.$$

Here for example  $A_{\mu} \leq H$  means that  $(A_{\mu}h, h)_{\mathcal{H}} \leq (Hh, h)_{\mathcal{H}}$ ,  $h \in \mathcal{H}$ . Let  $\mathcal{G}$  be a subspace of  $\mathcal{H}$  with finite codimension and let  $A$  be as above. Then the operators  $P_{\mathcal{G}}A_{\mu}|_{\mathcal{G}}$  and  $P_{\mathcal{G}}A_M|_{\mathcal{G}}$  are self-adjoint contractions which extend  $P_{\mathcal{G}}A|_{\mathcal{G}}$ , hence

$$(P_{\mathcal{G}}A|_{\mathcal{G}})_{\mu} \leq P_{\mathcal{G}}A_{\mu}|_{\mathcal{G}} \leq P_{\mathcal{G}}A_M|_{\mathcal{G}} \leq (P_{\mathcal{G}}A|_{\mathcal{G}})_M.$$

We give an example to show that the first and third equality can be strict. As  $A_{\mu} = -(-A)_M$  it suffices to give an example where  $P_{\mathcal{G}}A_M|_{\mathcal{G}} \neq (P_{\mathcal{G}}A|_{\mathcal{G}})_M$ : Let  $\mathcal{F}$  be a proper subspace of  $\mathcal{H}$  and let

$$A = -I_{\mathcal{F}} = \{\{f, -f\} \in \mathcal{H}^2 : f \in \mathcal{F}\}.$$

Then, as is shown in [11, Section 5],

$$A_{\mu} = -I_{\mathcal{H}} \quad \text{and} \quad A_M = -P_{\mathcal{F}} + P_{\mathcal{F}^{\perp}}.$$

Hence, since  $P_{\mathcal{G}}A|_{\mathcal{G}} = -I_{\mathcal{F} \cap \mathcal{G}}$ ,

$$(P_{\mathcal{G}}A|_{\mathcal{G}})_M = -P_{\mathcal{F} \cap \mathcal{G}} + P_{\mathcal{G} \ominus \mathcal{F} \cap \mathcal{G}}.$$

Now let

$\mathcal{H}$  be the Hilbert space  $\ell^2$  with orthonormal basis  $e_1, e_2, e_3, \dots$ ,

$\mathcal{F}$  be the subspace spanned by  $e_2, e_3, e_4, \dots$ ,

$\mathcal{G}$  be the subspace spanned by  $d_+, e_3, e_4, \dots$ ;  $d_+ := \frac{e_1 + e_2}{\sqrt{2}}$ .

Then  $\mathcal{F}^\perp$  and  $\mathcal{G}^\perp$  are spanned by  $e_1$  and  $d_- := (e_1 - e_2)/\sqrt{2}$ ,  $\mathcal{F} \cap \mathcal{G}$  is spanned by  $e_3, e_4, e_5, \dots$  and  $\mathcal{G} \ominus (\mathcal{F} \cap \mathcal{G})$  is spanned by  $d_+$ . We find that  $(P_{\mathcal{G}}A|_{\mathcal{G}})_M d_+ = d_+$ ,  $A_M d_+ = d_-$  and hence  $P_{\mathcal{G}}A_M d_+ = 0$ . (In fact in this example  $P_{\mathcal{G}}A_M|_{\mathcal{G}} = -P_{\mathcal{F} \cap \mathcal{G}}$ .) This proves the afore mentioned inequality.

## 5. Compressions of dilations

The following theorem is a special case of [12, Theorem 2.2]. We give a slightly different proof using reproducing kernel spaces.

**Theorem 5.1.** (i) Let  $S$  be a maximal symmetric linear relation in the Hilbert space  $\mathcal{K}$  with  $\mathbb{C}_- \subset \rho(S)$ , let  $\mathcal{H}$  be a subspace of  $\mathcal{K}$  and let  $R(\lambda) = P_{\mathcal{H}}R_S(\lambda)|_{\mathcal{H}}$ ,  $\lambda \in \rho(S) \cap \mathbb{C}_-$ . Then the kernel

$$K(\lambda, \mu) = \frac{R(\lambda) - R(\mu)^*}{\lambda - \mu^*} - R(\mu)^*R(\lambda)$$

is nonnegative on  $\mathbb{C}_-$ .

(ii) Conversely, let  $R(\lambda)$  be an analytic function on an open subset  $\Omega$  of  $\mathbb{C}_-$  whose values are bounded operators on a Hilbert space  $\mathcal{H}$ . If the kernel  $K(\lambda, \mu)$  is nonnegative on  $\Omega$ , then there exist a Hilbert space  $\mathcal{K}$ , which contains  $\mathcal{H}$  as a subspace, and a maximal symmetric linear relation  $S$  on  $\mathcal{K}$  with  $\mathbb{C}_- \subset \rho(S)$  such that

$$R(\lambda) = P_{\mathcal{H}}R_S(\lambda)|_{\mathcal{H}}, \quad \lambda \in \Omega.$$

Here  $\mathcal{K}$  and  $S$  can be chosen closely lower connected, which means that

$$\mathcal{K} = \overline{\text{span} \left( \mathcal{H} + \bigcup_{\lambda \in \mathbb{C}_-} R_S(\lambda)\mathcal{H} \right)}.$$

In this case  $\mathcal{K}$  and  $S$  are uniquely determined up to isomorphisms, which restricted to  $\mathcal{H}$  are equal to the identity operator on  $\mathcal{H}$ .

(iii) The Hilbert space  $\mathcal{K}$  and the linear relation  $S$  in (ii) can be chosen so that  $S$  is self-adjoint and  $\mathcal{K}$  and  $S$  are closely connected which means that

$$\mathcal{K} = \overline{\text{span} \left( \mathcal{H} + \bigcup_{\lambda \in \mathbb{C} \setminus \mathbb{R}} R_S(\lambda)\mathcal{H} \right)}$$

and implies that  $\mathcal{K}$  and  $S$  are uniquely determined up to isomorphisms, which restricted to  $\mathcal{H}$  are equal to the identity operator on  $\mathcal{H}$ .

**Proof.** (i) Since  $S$  is symmetric

$$\frac{R_S(\lambda) - R_S(\mu)^*}{\lambda - \mu^*} = R_S(\mu)^*R_S(\lambda), \quad \lambda, \mu \in \mathbb{C}_-,$$



and a straightforward calculation shows that for  $f, g \in \mathcal{H}$

$$(K(\lambda, \mu)f, g)_{\mathcal{H}} = ((I_{\mathcal{K}} - P_{\mathcal{H}})R_S(\lambda)f, R_S(\mu)g)_{\mathcal{K}}, \quad \lambda, \mu \in \mathbb{C}_-.$$

It follows that  $K(\lambda, \mu)$  is nonnegative.

(ii) Denote by  $\mathcal{N}$  the reproducing kernel Hilbert space with kernel  $K(\lambda, \mu)$ . Here we mean by this that  $\mathcal{N}$  is the completion of the linear span of the functions  $\mu \mapsto K(\lambda, \mu)f, \lambda \in \Omega, f \in \mathcal{H}$ , equipped with the inner product  $(\cdot, \cdot)_{\mathcal{N}}$  determined by

$$(K(\lambda, \cdot)f, K(\mu, \cdot)g)_{\mathcal{N}} = (K(\lambda, \mu)f, g)_{\mathcal{H}}, \quad \lambda, \mu \in \Omega, f, g \in \mathcal{H}.$$

Each element of  $\mathcal{N}$  is a function defined and anti holomorphic on  $\Omega$  with values in  $\mathcal{H}$ , and if  $f(\cdot) \in \mathcal{N}$ , then

$$(f(\cdot), K(\mu, \cdot)g)_{\mathcal{N}} = (f(\mu), g)_{\mathcal{H}}, \quad \mu \in \Omega, g \in \mathcal{H}.$$

We set  $\mathcal{K} = \mathcal{H} \oplus \mathcal{N}$  and define the linear relation  $S$  as the closure in  $\mathcal{K}^2$  of the linear relation

$$S_0 := \text{span} \{ \{R(\lambda)f + K(\lambda, \cdot)f, f + \lambda R(\lambda)f + \lambda K(\lambda, \cdot)f\} : \lambda \in \Omega, f \in \mathcal{H} \}.$$

If  $\lambda_1, \lambda_2, \dots, \lambda_l \in \Omega, f_1, f_2, \dots, f_l \in \mathcal{H}$ ,

$$\mathbf{f} := \sum_{j=1}^l R(\lambda_j)f_j + K(\lambda_j, \cdot)f_j \quad \text{and} \quad \mathbf{g} := \sum_{j=1}^l f_j + \lambda_j R(\lambda_j)f_j + \lambda_j K(\lambda_j, \cdot)f_j,$$

then  $\{\mathbf{f}, \mathbf{g}\} \in S_0$  and

$$(\mathbf{g}, \mathbf{f})_{\mathcal{K}} = \sum_{j,k}^l \left( \frac{\lambda_j R(\lambda_j) - \lambda_k^* R(\lambda_k)^*}{\lambda_j - \lambda_k^*} f_j, f_k \right)_{\mathcal{H}}.$$

Since the sum on the right hand side is real,  $S_0$  is symmetric, hence  $S$  is closed and symmetric and therefore  $\text{ran}(S - \lambda)$  is closed for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , see [16, Proposition 4.3]. Let  $\mu \in \Omega$ . Then

$$\text{ran}(S_0 - \mu) = \text{span} \{ f + (\lambda - \mu)R(\lambda)f + (\lambda - \mu)K(\lambda, \cdot)f : \lambda \in \Omega, f \in \mathcal{H} \}.$$

By choosing  $\lambda = \mu$ , it readily follows that  $\mathcal{H} \subset \text{ran}(S_0 - \mu)$  and from this it follows that all elements of the form  $K(\lambda, \cdot)f$  belong to  $\text{ran}(S_0 - \mu)$  for all  $f \in \mathcal{H}$  and all  $\lambda \in \Omega \setminus \{\mu\}$ . From the continuity of  $K(\lambda, \mu)$  and by considering closures we obtain  $\text{ran}(S - \mu) = \mathcal{K}$ . Since  $S$  is symmetric,  $\ker(S - \mu) = \{0\}$ , hence  $\mu \in \rho(S)$ . Since  $\mu \in \Omega$  is arbitrary, we have  $\Omega \subset \rho(S)$  and hence  $\mathbb{C}_- \subset \rho(S)$ . Moreover,  $S$  is maximal symmetric, because the upper defect index is 0:

$$\dim \ker(S^* - \nu) = \dim \text{ran}(S - \nu^*)^{\perp} = \dim \mathcal{K}^{\perp} = 0, \quad \nu \in \mathbb{C}_+.$$

From

$$(S - \lambda)^{-1}f = (S_0 - \lambda)^{-1}f = R(\lambda)f + K(\lambda, \cdot)f, \quad \lambda \in \Omega, f \in \mathcal{H},$$

it follows that  $P_{\mathcal{H}}R_S(\lambda)|_{\mathcal{H}} = R(\lambda)$  for all  $\lambda \in \Omega$ . This equality also implies that  $\mathcal{K}$  and  $S$  are closely lower connected. Assume that  $R(\lambda) = P_{\mathcal{H}}R_{S_1}(\lambda)|_{\mathcal{H}}$ ,  $\lambda \in \Omega$ , where  $S_1$  is a symmetric linear relation in a Hilbert space  $\mathcal{K}_1$  which extends  $\mathcal{H}$  and assume that  $\mathcal{K}_1$  and  $S_1$  are closely lower connected. Then for  $\lambda, \mu \in \Omega$  and  $f, g, h, k \in \mathcal{H}$

$$\begin{aligned} & (f + R_S(\lambda)g, h + R_S(\mu)k)_{\mathcal{K}} \\ &= (f, h)_{\mathcal{H}} + (R(\lambda)g, h)_{\mathcal{H}} + (f, R(\mu)k)_{\mathcal{H}} + (R(\lambda)g, R(\mu)k)_{\mathcal{H}} + (K(\lambda, \mu)g, k)_{\mathcal{H}} \\ &= (f + R_{S_1}(\lambda)g, h + R_{S_1}(\mu)k)_{\mathcal{K}_1}. \end{aligned}$$

By holomorphy the equality

$$(f + R_S(\lambda)g, h + R_S(\mu)k)_{\mathcal{K}} = (f + R_{S_1}(\lambda)g, h + R_{S_1}(\mu)k)_{\mathcal{K}_1}$$

holds for all  $\lambda, \mu \in \mathbb{C}_-$ . It follows that the mapping  $f + R_S(\lambda)g \mapsto f + R_{S_1}(\lambda)g$  can be extended to a linear isometry, and the lower connectedness of  $\mathcal{K}$  and  $S$  and the lower connectedness of  $\mathcal{K}_1$  and  $S_1$  imply that this isometry is a unitary mapping from  $\mathcal{K}$  onto  $\mathcal{K}_1$ , which we denote by  $U$ . Clearly  $Uf = f, f \in \mathcal{H}$ , and

$$S_1 = \{Uf, Ug\} : \{f, g\} \in S\}.$$

This completes the proof of (ii).

(iii) If  $S$  in (ii) is self-adjoint, then (iii) follows from (ii) and the proof of the theorem is finished. Assume  $S$  is not self-adjoint. Let  $\tilde{A}$  be any self-adjoint extension of  $S$  in a Hilbert space  $\tilde{\mathcal{K}}$ , that contains  $\mathcal{K}$  as a closed subspace. (For example let  $\tilde{A}$  be a self-adjoint extension of the direct sum of  $S$  and  $-S$  in the Hilbert space  $\mathcal{K}^2 \times \mathcal{K}^2$ , which exists because this direct sum is symmetric and has equal (possibly infinite) defect numbers.) Then  $R_{\tilde{A}}(\lambda)$  has the properties

$$R_{\tilde{A}}(\lambda)^* = R_{\tilde{A}}(\lambda^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (5.1)$$

$$R_{\tilde{A}}(\lambda) - R_{\tilde{A}}(\mu) = (\lambda - \mu)R_{\tilde{A}}(\lambda)R_{\tilde{A}}(\mu), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad (5.2)$$

$$R_{\tilde{A}}(\lambda)(S - \lambda) \subset I, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (5.3)$$

The inclusions (5.3) and  $\mathbb{C}_- \subset \rho(S)$  imply that

$$P_{\mathcal{K}}R_{\tilde{A}}(\lambda)|_{\mathcal{K}} = R_S(\lambda), \quad \lambda \in \mathbb{C}_-, \quad (5.4)$$

and hence, by (ii),

$$P_{\mathcal{H}}R_{\tilde{A}}(\lambda)|_{\mathcal{H}} = R(\lambda), \quad \lambda \in \Omega. \quad (5.5)$$

From (5.4) and (5.1) it follows that the function  $P_{\mathcal{K}}R_{\tilde{A}}(\lambda)|_{\mathcal{K}}$  on  $\mathbb{C} \setminus \mathbb{R}$  is uniquely determined by the resolvent  $R_S(\lambda)$  on  $\mathbb{C}_-$  of the maximal symmetric linear relation  $S$  (In [1, Section 112] this was shown for densely defined maximal symmetric operators.), but the self-adjoint extension  $\tilde{A}$  need not be unique. We can make it unique up to isomorphisms by removing its invariant part in the space  $\tilde{\mathcal{K}} \ominus \mathcal{K}$ . To show this, consider the subspace

$$\hat{\mathcal{K}} = \overline{\text{span}(\mathcal{K} + \cup_{\lambda \in \mathbb{C} \setminus \mathbb{R}} R_{\tilde{A}}(\lambda)\mathcal{K})} \subset \tilde{\mathcal{K}}$$

and define for a fixed  $\tau \in \mathbb{C} \setminus \mathbb{R}$  the linear relation

$$\hat{A} = \{R_{\tilde{A}}(\tau)k, k + \tau R_{\tilde{A}}(\tau)k\} : k \in \mathcal{K} \} \subset \hat{\mathcal{K}}^2.$$

Then, by (5.1) and (5.2),  $R_{\widehat{A}}(\lambda)$ ,  $\lambda \in \mathbb{C}$ , maps  $\widehat{\mathcal{K}}$  to itself,  $\widehat{A}$  is independent of  $\tau \in \mathbb{C} \setminus \mathbb{R}$  and is a self-adjoint linear relation in  $\widehat{\mathcal{K}}$  (see for example [15, Theorem 4.1]) and the resolvent  $R_{\widehat{A}}(\lambda)$  of  $\widehat{A}$  satisfies

$$R_{\widehat{A}}(\lambda) = R_{\widehat{A}}(\lambda)|_{\widehat{\mathcal{K}}}, \quad \lambda \in \mathbb{C}.$$

From (5.4) and (5.5) it follows that

$$P_{\mathcal{H}}R_{\widehat{A}}(\lambda)|_{\mathcal{H}} = R(\lambda), \quad \lambda \in \Omega.$$

Item (iii) now follows if we take  $\widehat{\mathcal{K}}$  and  $\widehat{A}$  for  $\mathcal{K}$  and  $S$ . We leave the proofs of the remaining details to the reader.  $\square$

**Remark 5.2.** It follows from Theorem 5.1 (iii) that if  $R(\lambda)$  is a bounded operator function for which the kernel  $K(\lambda, \mu)$  is nonnegative on an open subset  $\Omega \subset \mathbb{C}_-$ , then the function

$$\widetilde{R}(\lambda) = \begin{cases} R(\lambda^*)^*, & \lambda \in \Omega^* := \{\mu^* : \mu \in \Omega\}, \\ R(\lambda), & \lambda \in \Omega, \end{cases}$$

and the kernel  $K(\lambda, \mu)$  associated with it on  $\Omega^* \cup \Omega$  can be extended to a bounded operator function on  $\mathbb{C} \setminus \mathbb{R}$  which is symmetric with respect to the real axis and a nonnegative kernel on  $\mathbb{C} \setminus \mathbb{R}$ . This is a special case of [12, Theorem 2.2].

Let  $T$  be a linear relation in a Hilbert space  $\mathcal{H}$  with  $\rho(T) \neq \emptyset$ . A linear relation  $A$  in a Hilbert space  $\mathcal{K}$  is called a dilation of  $T$  if  $\mathcal{H}$  is a subspace of  $\mathcal{K}$ ,  $\rho(T) \cap \rho(A) \neq \emptyset$  and

$$R_T(\lambda) = P_{\mathcal{H}}R_A(\lambda)|_{\mathcal{H}}, \quad \lambda \in \rho(T) \cap \rho(A).$$

The dilation  $A$  of  $T$  is called minimal if

$$\mathcal{K} = \overline{\text{span}(\mathcal{H} + \bigcup_{\lambda \in \rho(A)} R_A(\lambda)\mathcal{H})}.$$

If  $T$  is a maximal dissipative linear relation in a Hilbert space  $\mathcal{H}$ , then, by Lemma 2.3,  $\mathbb{C}_- \subset \rho(T)$  and  $K_T(\lambda, \mu)$  is nonnegative on  $\mathbb{C}_-$ . Parts (ii) and (iii) of Theorem 5.1 imply that  $T$  has a minimal self-adjoint dilation, which is uniquely determined up to isomorphisms. If  $T$  is a densely defined maximal dissipative operator this is well-known. For explicit constructions and further details we refer the reader to [26, Chapter 4]. We now come to the main theorem of this section.

**Theorem 5.3.** If  $A$  in a Hilbert space  $\mathcal{K}$  is the minimal self-adjoint dilation of a maximal dissipative linear relation  $T$  in the Hilbert space  $\mathcal{H}$  and if  $\mathcal{G}$  is a subspace of  $\mathcal{H}$  with finite codimension, then the compression  $P_{(\mathcal{K} \ominus \mathcal{H}) \oplus \mathcal{G}} A|_{(\mathcal{K} \ominus \mathcal{H}) \oplus \mathcal{G}}$  is the minimal self-adjoint dilation of the compression  $P_{\mathcal{G}} T|_{\mathcal{G}}$ , which is maximal dissipative in  $\mathcal{G}$ .

The proof of this theorem follows directly from Theorem 3.1, Theorem 4.5 and the next lemma.

**Lemma 5.4.** Let  $T$  be a linear relation in a Hilbert space  $\mathcal{H}$  and let  $A$  be a linear relation in a Hilbert space  $\mathcal{K}$ , which contains  $\mathcal{H}$  as a subspace. Assume there is a nonempty open subset  $O \subset \rho(T) \cap \rho(A)$  such that

$$R_T(\lambda) = P_{\mathcal{H}}R_A(\lambda)|_{\mathcal{H}}, \quad \lambda \in O.$$

Let  $\mathcal{G}$  be subspace of  $\mathcal{H}$  with finite codimension  $m$  and let  $\mathfrak{B} = (b_1 \ b_2 \ \cdots \ b_m)$  be a row vector whose  $m$  entries  $b_j$  form a basis of  $\mathcal{H} \ominus \mathcal{G}$ . Assume that the open set

$$U := \{\lambda \in \mathbb{C} : \det (R_T(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}} \neq 0\}$$

is nonempty. If  $T_0 = P_{\mathcal{G}}T|_{\mathcal{G}}$  and  $A_0 = P_{\mathcal{G} \oplus \mathcal{N}}A|_{\mathcal{G} \oplus \mathcal{N}}$ ,  $\mathcal{N} = \mathcal{K} \ominus \mathcal{H}$ , then

$$R_{T_0}(\lambda) = P_{\mathcal{G}}R_{A_0}(\lambda)|_{\mathcal{G}}, \quad \lambda \in U.$$

**Proof.** Since  $U = \{\lambda \in \mathbb{C} : \det (R_A(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{K}} \neq 0\}$ , we may apply Lemma 3.2 to obtain

$$R_{A_0}(\lambda) = R_A(\lambda)|_{\mathcal{G} \oplus \mathcal{N}} - R_A(\lambda)\mathfrak{B} (R_A(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{K}}^{-1} (R_A(\lambda)|_{\mathcal{G} \oplus \mathcal{N}} \cdot, \mathfrak{B})_{\mathcal{K}}.$$

Hence

$$\begin{aligned} P_{\mathcal{G}}R_{A_0}(\lambda)|_{\mathcal{G}} &= P_{\mathcal{G}}R_T(\lambda)|_{\mathcal{G}} - P_{\mathcal{G}}R_T(\lambda)\mathfrak{B} (R_T(\lambda)\mathfrak{B}, \mathfrak{B})_{\mathcal{H}}^{-1} (R_T(\lambda)|_{\mathcal{G}} \cdot, \mathfrak{B})_{\mathcal{H}} \\ &= P_{\mathcal{G}}R_{T_0}(\lambda)|_{\mathcal{G}} = R_{T_0}(\lambda). \quad \square \end{aligned}$$

## 6. Appendix

Here we recall the vector notation from [10, p. 477]. Let

$$\mathfrak{F} = (f_1 \ f_2 \ \cdots \ f_m) \tag{6.1}$$

be a row vector whose  $m$  entries  $f_j$  are elements of a Hilbert or Krein space  $(\mathcal{K}, (\cdot, \cdot)_{\mathcal{K}})$ . If  $B$  is a bounded operator in  $\mathcal{K}$  we define the row vector  $B\mathfrak{F}$  by

$$B\mathfrak{F} = (Bf_1 \ Bf_2 \ \cdots \ Bf_m)$$

and if  $E = (e_{jk})_{j=1, \dots, m; k=1, \dots, l}$  is an  $m \times l$  matrix, then  $\mathfrak{F}E$  is the  $1 \times l$  vector

$$\mathfrak{F}E = \left( \sum_{j=1}^m e_{j1}f_j \ \sum_{j=1}^m e_{j2}f_j \ \cdots \ \sum_{j=1}^m e_{jl}f_j \right).$$

Thus for  $x = (x_1 \ x_2 \ \cdots \ x_m)^{\top} \in \mathbb{C}^m$  we have  $\mathfrak{F}x = \sum_{j=1}^m x_j f_j$ . Let

$$\mathfrak{G} = (g_1 \ g_2 \ \cdots \ g_l)$$

be a row vector of  $l$  elements  $g_j$  from  $\mathcal{K}$ . Then  $(\mathfrak{F}, \mathfrak{G})_{\mathcal{K}}$  stands for the  $l \times m$  matrix

$$(\mathfrak{F}, \mathfrak{G})_{\mathcal{K}} := (\gamma_{jk})_{j=1, \dots, l; k=1, \dots, m} \quad \text{with } \gamma_{jk} = (f_k, g_j)_{\mathcal{K}}.$$

It follows that  $(\mathfrak{F}, \mathfrak{G})_{\mathcal{K}}^* = (\mathfrak{G}, \mathfrak{F})_{\mathcal{K}}$  and that if  $E$  and  $D$  are matrices over  $\mathbb{C}$  with  $m$  and  $l$  rows, respectively, then

$$(\mathfrak{F}E, \mathfrak{G}D)_{\mathcal{K}} = D^*(\mathfrak{F}, \mathfrak{G})_{\mathcal{K}}E. \tag{6.2}$$

In particular,  $(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}$  is the  $m \times m$  Gram matrix of  $\mathfrak{F}$ . If the entries of  $\mathfrak{F}$  span a Krein (= non-degenerated) subspace  $\mathcal{F}$  of  $\mathcal{K}$ , then  $(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}$  is invertible if and only if the entries of  $\mathfrak{F}$  are linearly independent and in this case the projection  $P_{\mathcal{F}}$  onto  $\mathcal{F}$  is given by

$$P_{\mathcal{F}} = \mathfrak{F}(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}^{-1}(\cdot, \mathfrak{F})_{\mathcal{K}}. \quad (6.3)$$

In the vector notation the “if” part follows from (6.2) and the implications

$$\begin{aligned} x \in \mathbb{C}^m \text{ and } (\mathfrak{F}, \mathfrak{F})_{\mathcal{K}} x = 0 &\Rightarrow y^* (\mathfrak{F}, \mathfrak{F})_{\mathcal{K}} x = 0 \quad \forall y \in \mathbb{C}^m \\ &\Rightarrow (\mathfrak{F}x, \mathfrak{F}y)_{\mathcal{K}} = 0 \quad \forall y \in \mathbb{C}^m \Rightarrow \mathfrak{F}x \in \mathcal{F} \cap \mathcal{F}^{\perp} = \{0\} \Rightarrow x = 0, \end{aligned}$$

and for example the equality  $P_{\mathcal{F}}^2 = P_{\mathcal{F}}$  follows from (6.2) and the equalities

$$\begin{aligned} P_{\mathcal{F}}^2 g &= \mathfrak{F}(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}^{-1} (P_{\mathcal{F}} g, \mathfrak{F})_{\mathcal{K}} = \mathfrak{F}(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}^{-1} (\mathfrak{F}(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}^{-1} (g, \mathfrak{F})_{\mathcal{K}}, \mathfrak{F})_{\mathcal{K}} \\ &= \mathfrak{F}(\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}^{-1} (\mathfrak{F}, \mathfrak{F})_{\mathcal{K}} (\mathfrak{F}, \mathfrak{F})_{\mathcal{K}}^{-1} (g, \mathfrak{F})_{\mathcal{K}} = P_{\mathcal{F}} g, \quad g \in \mathcal{F}. \end{aligned}$$

The equality (6.2) is trivial in case  $\mathcal{K} = L^2(\mathbb{R})$ , for then

$$(\mathfrak{F}, \mathfrak{G})_{L^2(\mathbb{R})} = \left( \int_{\mathbb{R}} g_j(t)^* f_k(t) dt \right)_{j=1, \dots, l; k=1, \dots, m} = \int_{\mathbb{R}} \mathfrak{G}^*(t) \mathfrak{F}(t) dt,$$

where for instance  $\mathfrak{F}(t) = (f_1(t) f_2(t) \cdots f_m(t))$  with  $f_j \in L^2(\mathbb{R})$ ,  $j = 1, 2, \dots, m$ . For example, it follows from the integral formula

$$\frac{i}{z - w^*} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{t - w^*} \frac{1}{t - z} dt, \quad z, w \in \mathbb{C}_+,$$

that the Gram matrix of the vector function

$$\mathfrak{F}(t) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{t - z_1} \quad \frac{1}{t - z_2} \quad \cdots \quad \frac{1}{t - z_m} \right), \quad z_1, z_2, \dots, z_m \in \mathbb{C}_+,$$

is given by

$$(\mathfrak{F}, \mathfrak{F})_{L^2(\mathbb{R})} = \left( \frac{i}{z_k - z_j^*} \right)_{j,k=1}^m. \quad (6.4)$$

It follows that the matrix  $\left( i / (z_k - z_j^*) \right)_{j,k=1}^m$  is nonnegative. This is used in the proof of Lemma 2.3 (d). As last example we recall the following well-known lemma [19, Lemma 2.1] (see also [18, Lemma IV.2.8]) and repeat its proof using the vector notation. Part (ii) is used in the proof of Theorem 4.7.

**Lemma 6.1.** *Let  $\mathcal{G}$  be a Krein subspace of a Krein space  $\mathcal{K}$  with finite codimension  $m$  and let  $\mathcal{D}$  be a dense linear subset of  $\mathcal{K}$ . Then*

- (i)  $\mathcal{G} \cap \mathcal{D}$  is dense in  $\mathcal{G}$  and
- (ii) *there is subspace  $\mathcal{F} \subset \mathcal{D}$  with  $\dim \mathcal{F} = m$  such that  $\mathcal{K} = \mathcal{G} + \mathcal{F}$ , direct sum.*

**Proof.** Clearly  $m = \dim \mathcal{G}^{\perp}$ . Let  $\mathfrak{B}$  be a  $1 \times m$  vector whose entries form a basis for  $\mathcal{G}^{\perp}$ . Then the  $m \times m$  matrix  $(\mathfrak{B}, \mathfrak{B})_{\mathcal{K}}$  is invertible. Since the inner product is continuous and  $\mathcal{D}$  is dense in  $\mathcal{K}$ , there is a  $1 \times m$  vector  $\mathfrak{F}_0$  with entries from  $\mathcal{D}$  such that the  $m \times m$  matrix  $(\mathfrak{F}_0, \mathfrak{B})_{\mathcal{K}}$  is invertible. Set  $\mathfrak{F} = \mathfrak{F}_0(\mathfrak{F}_0, \mathfrak{B})_{\mathcal{K}}^{-1}$ . Then  $\mathfrak{F}$  is a  $1 \times m$  vector with entries from  $\mathcal{D}$  and, by (6.2),  $(\mathfrak{F}, \mathfrak{B})_{\mathcal{K}} = (\mathfrak{F}_0, \mathfrak{B})_{\mathcal{K}}(\mathfrak{F}_0, \mathfrak{B})_{\mathcal{K}}^{-1} = I$ .

(i) Let  $x \in \mathcal{G}$ . As  $\mathcal{D}$  is dense in  $\mathcal{K}$ , there is a sequence  $x_n$  in  $\mathcal{D}$  such that  $x_n \rightarrow x$ . Set  $\tilde{x}_n := x_n - \mathfrak{F}(x_n, \mathfrak{B})_{\mathcal{K}}$ . Then  $\tilde{x}_n$  belongs to  $\mathcal{D}$ . It also belongs to  $\mathcal{G}$  because, by (6.2),

$$(\tilde{x}_n, \mathfrak{B})_{\mathcal{K}} = (x_n, \mathfrak{B})_{\mathcal{K}} - (\mathfrak{F}, \mathfrak{B})_{\mathcal{K}}(x_n, \mathfrak{B})_{\mathcal{K}} = 0,$$

which shows that  $\tilde{x}_n$  is orthogonal to the entries of  $\mathfrak{B}$ , hence  $\tilde{x}_n \in (\mathcal{G}^{\perp})^{\perp} = \mathcal{G}$ . Since  $(x, \mathfrak{B})_{\mathcal{K}} = 0$ , we have  $\tilde{x}_n \rightarrow x - \mathfrak{F}(x, \mathfrak{B})_{\mathcal{K}} = x$ . This implies (i).

(ii) Let  $\mathcal{F}$  be the span of the  $m$  entries of  $\mathfrak{F}$ . Then  $\mathcal{F} \subset \mathcal{D}$  and the equality  $(\mathfrak{F}, \mathfrak{B})_{\mathcal{K}} = I$  implies that  $\dim \mathcal{F} = m$  and  $\mathcal{F} \cap \mathcal{G} = \{0\}$ . For each  $h \in \mathcal{K}$  we have  $\mathfrak{F}(h, \mathfrak{B})_{\mathcal{K}} \in \mathcal{F}$  and  $h - \mathfrak{F}(h, \mathfrak{B})_{\mathcal{K}} \in \mathcal{G}$ . Hence  $\mathcal{K} = \mathcal{G} + \mathcal{F}$ . This implies (ii).  $\square$

For  $f \in \mathcal{K}$  and  $\mathfrak{F}$  as in (6.1) we denote by  $(\mathfrak{F} : f)$  the  $1 \times (k+1)$  vector

$$(\mathfrak{F} : f) = (f_1 f_2 \cdots f_k f). \quad (6.5)$$

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